



12.2 Eigenvalues and eigenvectors of signals

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Introduction

- In this topic, we will
 - Examine the advance operator and differential operator
 - Contrast their behavior with the delay operator and the anti-derivative operator
 - Find eigenvectors for the first two
 - We will also find null spaces of the first two, as well
 - Observe how we can use these eigenvectors to solve various equations





Discrete-time signals

- Consider the vector space of discrete-time signals:

$$x = (x[0], x[1], x[2], x[3], \dots)$$

- The delay operator is defined as

$$Dx = (0, x[0], x[1], x[2], \dots)$$

- It does not have any eigenvectors or eigenvalues, because if

$$Dx = \lambda x,$$

then

$$(0, x[0], x[1], x[2], \dots) = \lambda (x[0], x[1], x[2], x[3], \dots)$$

- Now, if x is not the zero signal, neither is Dx , so D does not have a non-trivial null space
- If $\lambda \neq 0$, then this is true if and only if x is the zero signal





Discrete-time signals

- Still considering the vector space of discrete-time signals:

$$x = (x[0], x[1], x[2], x[3], \dots)$$

- The advance operator defined as

$$Ex = (x[1], x[2], x[3], x[4], \dots)$$

- Thus has a real (or complex) value λ as an eigenvalue

$$Ex = \lambda x,$$

if

$$(x[1], x[2], x[3], x[4], \dots) = \lambda (x[0], x[1], x[2], x[3], \dots)$$





Discrete-time signals

- Now, if $\lambda = 0$, then

$$(x[1], x[2], x[3], x[4], \dots) = (0, 0, 0, 0, \dots)$$

- Thus, an eigenvector corresponding to the eigenvalue $\lambda = 0$ is

$$x = (1, 0, 0, 0, \dots)$$

- Now, if $\lambda \neq 0$, then

$$(x[1], x[2], x[3], x[4], \dots) = (\lambda x[0], \lambda x[1], \lambda x[2], \lambda x[3], \dots)$$

- Thus, if $x[0] = 1$, then

$$x[1] = \lambda$$

$$x[2] = \lambda^2$$

$$x[3] = \lambda^3$$

- Thus, an eigenvector corresponding to the eigenvalue λ is

$$(1, \lambda, \lambda^2, \lambda^3, \lambda^4, \dots)$$

- That is, the geometric sequence are the eigenvectors





Discrete-time signals

- Recall that in finite-dimensional vector spaces,
if $A:\mathbf{R}^n \rightarrow \mathbf{R}^n$ then A has at most n eigenvalues
- For every real number γ , there is an eigenvector with that value γ as an eigenvalue
 - Thus, there are infinitely many eigenvalues, and as many eigenvalues as there are real numbers
 - Also, the dimension of the eigenspace corresponding to any eigenvalue is one, so given γ and $(1, \gamma, \gamma^2, \gamma^3, \gamma^4, \dots)$, the eigenvectors of γ are all scalar multiples of the corresponding geometric sequence
- If we consider complex geometric sequences,
there is an eigenvector for every complex value γ





Discrete-time signals

- First, let us find solutions to

$$Dx = x_0$$

- Thus, if $x_0 = (x_0[0], x_0[1], x_0[2], x_0[3], \dots)$, it follows that all solutions are of the form

$$x = (c, x_0[0], x_0[1], x_0[2], x_0[3], \dots)$$

- Here, c is an arbitrary scalar value
- Thus, all solutions are of the form

$$(0, x_0[0], x_0[1], x_0[2], x_0[3], \dots) + c(1, 0, 0, 0, 0, \dots)$$



This is in the null space of E and $ED = I$





Discrete-time signals

- Next, let us find solutions to

$$Ex + \alpha_0 x = 0$$

- Substituting our eigensignals into this

$$E\gamma^n + \alpha_0 \gamma^n = 0$$

$$\gamma \gamma^n + \alpha_0 \gamma^n = 0$$

$$(\gamma + \alpha_0) \gamma^n = 0$$

- Thus, the ratio must be $\gamma = -\alpha_0$





Discrete-time signals

- For example, given $Ex + 2x = 0$, we note that if $x[n] = (-2)^n$ then $Ex = -2x$ and

$$Ex + 2x = 0$$

$$-2x + 2x = 0$$

- To view this more clearly:

$$\begin{array}{r} 2(\quad 1, \quad -2, \quad 4, \quad -8, \quad 16, \quad -32, \quad \dots) \\ + (-2, \quad 4, \quad -8, \quad 16, \quad -32, \quad 64, \quad \dots) \\ \hline (\quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad \dots) \end{array}$$





Discrete-time signals

- Next, let us find solutions to

$$E^2 x + \alpha_1 E x + \alpha_0 x = 0$$

- Substituting our eigensignals into this

$$E^2 \gamma^n + \alpha_1 E \gamma^n + \alpha_0 \gamma^n = 0$$

$$\gamma^2 \gamma^n + \alpha_1 \gamma \gamma^n + \alpha_0 \gamma^n = 0$$

$$(\gamma^2 + \alpha_1 \gamma + \alpha_0) \gamma^n = 0$$

- Thus, the ratio γ must be a root of the quadratic





Discrete-time signals

- For example, given $E^2x + 7Ex + 12x = 0$
 - This requires $\gamma^2 + 7\gamma + 12 = 0$, so $\gamma = -3$ or $\gamma = -4$
 - We note that if $x[n] = (-3)^n$ then $Ex = -3x$ and $E^2x = 9x$, so
- Similarly, note that if $x[n] = (-4)^n$ then $Ex = -4x$ and $E^2x = 16x$, so

$$\begin{aligned}E^2x + 7Ex + 12x &= 0 \\ 9x + 7(-3)x + 12x &= 0\end{aligned}$$

$$\begin{aligned}E^2x + 7Ex + 12x &= 0 \\ 16x + 7(-4)x + 12x &= 0\end{aligned}$$





Discrete-time signals

- Again, to view this more clearly:

$$\begin{array}{r} 12(\quad 1, \quad -3, \quad 9, \quad -27, \quad 81, \quad \dots) \\ 7(\quad -3, \quad 9, \quad -27, \quad 81, \quad 243, \quad \dots) \\ + (\quad 9, \quad -27, \quad 81, \quad 243, \quad -729, \quad \dots) \\ \hline (\quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad \dots) \end{array}$$





Continuous-time signals

- Consider the space of continuous-time signals:

$$y(t)$$

- You are already familiar with the linear differential operator:

$$\frac{d}{dt} y(t)$$

- Now, for y to be an eigenfunction of this operator, we require

$$\frac{d}{dt} y(t) = \lambda y(t)$$

- You may have already guessed such solutions:

$$\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$$





Continuous-time signals

- As before, for every real number λ , there is an eigenvector $e^{\lambda t}$ with that value λ as an eigenvalue
 - Thus, again, there are infinitely many eigenvalues, and as many eigenvalues as there are real numbers
 - Also, the dimension of the eigenspace corresponding to any eigenvalue is one, so given λ and $e^{\lambda t}$, the eigenvectors of λ are all scalar multiples of the corresponding exponential function
- If we consider complex-valued functions, there is an eigenvector for every complex value λ





Continuous-time signals

- First, let us find solutions to

$$\frac{d}{dt} y(t) = y_0(t)$$

- We are finding the anti-derivative of the function $y_0(t)$
 - From calculus, you know that if $Y_0(t)$ is one anti-derivative, then all anti-derivatives are of the form $Y_0(t) + c$
 - Here, c is an arbitrary constant
- You will remember that the constant functions form the null space of the differential operator
 - Just like the signal $(1, 0, 0, 0, \dots)$ form the null space of the advance operator





Continuous-time signals

- Next, let us find solutions to

$$\frac{d}{dt} y(t) + \alpha_0 y(t) = 0$$

- Substituting our eigenfunctions into this

$$\frac{d}{dt} e^{\lambda t} + \alpha_0 e^{\lambda t} = 0$$

$$\lambda e^{\lambda t} + \alpha_0 e^{\lambda t} = 0$$

$$(\lambda + \alpha_0) e^{\lambda t} = 0$$

- Thus, the scalar in the exponent must be $\lambda = -\alpha_0$





Continuous-time signals

- For example,

$$\frac{d}{dt}y(t) + 2y(t) = 0$$

we note that if $y(t) = e^{-2t}$, then

$$\frac{d}{dt}e^{-2t} = -2e^{-2t}$$

- Thus, we note that

$$\frac{d}{dt}e^{-2t} + 2e^{-2t} = 0$$

$$-2e^{-2t} + 2e^{-2t} = 0$$





Continuous-time signals

- Next, let us find solutions to

$$\frac{d^2}{dt^2} y(t) + \alpha_1 \frac{d}{dt} y(t) + \alpha_0 y(t) = 0$$

- Substituting our eigenfunctions into this

$$\frac{d^2}{dt^2} e^{\lambda t} + \alpha_1 \frac{d}{dt} e^{\lambda t} + \alpha_0 e^{\lambda t} = 0$$

$$\lambda^2 e^{\lambda t} + \alpha_1 \lambda e^{\lambda t} + \alpha_0 e^{\lambda t} = 0$$

$$(\lambda^2 + \alpha_1 \lambda + \alpha_0) e^{\lambda t} = 0$$

- Thus, the scalar λ must be a root of the quadratic





Continuous-time signals

- For example,

$$\frac{d^2}{dt^2} y(t) + 7 \frac{d}{dt} y(t) + 12y(t) = 0$$

- This requires $\lambda^2 + 7\lambda + 12 = 0$, so $\lambda = -3$ or $\lambda = -4$

- Note that if $y(t) = e^{-3t}$, then $\frac{d}{dt} y(t) = -3y(t)$ and $\frac{d^2}{dt^2} y(t) = 9y(t)$, so

$$\frac{d^2}{dt^2} e^{-3t} + 7 \frac{d}{dt} e^{-3t} + 12e^{-3t} = 0$$

$$9e^{-3t} + 7(-3)e^{-3t} + 12e^{-3t} = 0$$

- Similar results for $y(t) = e^{-4t}$,





Summary

- Following this topic, you now
 - Understand that the advance operator and the differential operator have null spaces
 - Understand that the delay operator and the anti-derivative operator have infinitely many solutions, and those include adding all scalar multiples of the null spaces of the corresponding operators
 - Know that the geometric sequences are the eigensignals of the advance operator
 - Know that the exponential functions are the eigenfunctions of the differential operator
 - Have a reinforced understanding that an operator applied to an eigenvector results in a scalar multiple of that eigenvector





References

- [1] <https://en.wikipedia.org/wiki/Eigenfunction>





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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